Euclidean Domain :- An integral Domain R is

Let $a \neq 0$, $a \in R$ define $\delta(a)$ or $d(a) > 0$

s.t. i) $\forall$ $a, b \in R$, $a \neq 0$, $b \neq 0$, $\delta(ab) \geq \delta(a)$

ii) $\forall$ $a, b \in R$, $b \neq 0$ $\exists$ $q, r \in R$

s.t. $a = bq + r$, either $r = 0$ or $\delta(r) < \delta(b)$

δ or d or ϕ is called Euclidean Domain or R.

Example 1. Prove that $Z[i]$ the ring of Gaussian integers is a E.D.

Sol :- We know $Z[i] = \{a+ib \,;\, a, b \in Z\}$ is integral Domain

$$d(a+ib) = a^2+b^2 > 0$$

$$d(a+ib) > 0$$

i) Let $\alpha, \beta \in Z[i]$ T.P $d(\alpha\beta) \geq d(\alpha)$

$\alpha = a+ib$

$\beta = c+id$

$$d(\alpha\beta) = d((a+ib)(c+id))$$

$$= d[ac + aid + ibc + i^2bd]$$

$$= d[(ac-bd) + i(bc+ad)]$$

$$= (ac-bd)^2 + (bc+ad)^2$$

$$= a^2c^2 + b^2d^2 - \cancel{2abcd} + b^2c^2 + a^2d^2 + \cancel{2abcd}$$

$$d(\alpha\beta) = a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2$$

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$$d(\alpha\beta) = (a^2+b^2)(c^2+d^2) \quad - \text{①}$$

We have $\beta = c+id$

$\beta \neq 0 \quad \therefore \quad c \neq 0 \text{ or } d \neq 0$

$d(\beta) = c^2+d^2 \geq 1$

$d(\beta) \geq 1$

By ① $\quad d(\alpha\beta) = d(\alpha)\, d(\beta)$

$d(\alpha\beta) \geq d(\alpha)$

ii) $\underline{\underline{T.P}}$ Let $\alpha, \beta, \gamma, \delta \in Z[i]$

$\alpha = \beta\gamma + \delta \quad \Rightarrow \quad \delta = \alpha - \beta\gamma$

$\underline{\underline{T.P}} \quad d(\delta) < d(\beta)$

Let $\alpha = a+ib$, $\beta = c+id$, $\gamma = e+if$

$a, b, c, d, e, f \in Z$

$$d(\delta) = d(\alpha - \beta\gamma)$$

$$= d\left(\beta\left(\frac{\alpha}{\beta} - \gamma\right)\right) \quad - \text{②}$$

Now $\dfrac{\alpha}{\beta} = \dfrac{a+ib}{c+id} \times \dfrac{c-id}{c-id}$

$$= \frac{ac - iad + cib - i^2bd}{c^2 - i^2d^2}$$

$$= \frac{ac - iad + ibc + bd}{c^2+d^2}$$

$$\frac{\alpha}{\beta} = \frac{ac+bd}{c^2+d^2} + i\,\frac{bc-ad}{c^2+d^2} \Rightarrow e' + if'$$

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where $e' = \dfrac{ac+bd}{c^2+d^2}$, $f' = \dfrac{bc-ad}{c^2+d^2}$

By (2) $d(\delta) = d(\beta(e'+if') - \gamma]$

$= d(\beta)\, d[e'+if' - (e+if)]$

$= d(\beta)\, d[(e'-e) + i(f'-f)]$

$d(\delta) = d(\beta)[(e'-e)^2 + (f'-f)^2]$

We have $e'-e \leq \frac{1}{2}$

$f'-f \leq \frac{1}{2}$

$\therefore\ d(\delta) \leq d(\beta)\ [(\frac{1}{2})^2 + (\frac{1}{2})^2]$

$= d(\beta)[\frac{1}{4} + \frac{1}{4}]$

$= d(\beta)\ [\frac{2}{4}]$

$= d(\beta)\ (\frac{1}{2})$

$d(\delta) \leq \dfrac{d(\beta)}{2}$

$\boxed{d(\delta) < d(\beta)}$ $[\because \frac{1}{2} < 1]$

Hence Proved

$Z[i]$ is E.D

example 2. Show that $Z[\sqrt{2}] = \{a+\sqrt{2}b \,;\, a,b \in Z\}$ is a E.D.

Sol:

We know $Z[\sqrt{2}]$ is an integral Domain.

Let $\alpha = a+\sqrt{2}b$

$$d(\alpha) = |a^2-2b^2| \geq 0$$

i) Let $\alpha = a+\sqrt{2}b$, $\beta = c+\sqrt{2}d \in Z[\sqrt{2}]$

T.P $d(\alpha\beta) \geq d(\alpha)$

$$d(\alpha\beta) = d((a+\sqrt{2}b)(c+\sqrt{2}d))$$

$$= d[ac+\sqrt{2}ad+\sqrt{2}cb+2bd]$$

$$= d[(ac+2bd)+\sqrt{2}(ad+bc)]$$

$$= (ac+2bd)^2 - 2(ad+bc)^2$$

$$= (a^2c^2+4b^2d^2+4abcd) - 2(a^2d^2+b^2c^2+2abcd)$$

$$= a^2c^2+4b^2d^2+\cancel{4abcd}-2a^2d^2-2b^2c^2-\cancel{4abcd}$$

$$= a^2c^2-2b^2c^2+4b^2d^2-2a^2d^2$$

$$= c^2(a^2-2b^2) - 2d^2(a^2-2b^2)$$

$$d(\alpha\beta) = (c^2-2d^2)(a^2-2b^2) \quad - ①$$

We have $\beta = c+\sqrt{2}d$, $\beta \neq 0$

$c+\sqrt{2}d \neq 0$, $c \neq 0$ or $d \neq 0$

$$d(\beta) = c^2-2d^2 \geq 1$$

$$d(\beta) \geq 1$$

By ① $d(\alpha\beta) = d(\alpha)\,d(\beta)$

$$d(\alpha\beta) \geq d(\alpha)$$

ii) Let $\alpha, \beta, \gamma, \delta \in \mathbb{Z}[i]$

$\alpha = \beta\gamma + \delta$

T.P $d(\delta) < d(\beta)$

Let $\alpha = a + \sqrt{2}b$, $\beta = c + \sqrt{2}d$, $\gamma = e + \sqrt{2}f$

$d(\delta) = d(\alpha - \beta\gamma)$

$= d\left(\beta\left(\frac{\alpha}{\beta} - \gamma\right)\right)$ — ②

Now $\frac{\alpha}{\beta} = \frac{a + \sqrt{2}b}{c + \sqrt{2}d} \times \frac{c - \sqrt{2}d}{c - \sqrt{2}d}$

$= \frac{ac - \sqrt{2}ad + \sqrt{2}bc - 2bd}{c^2 - 2d^2}$

$= \frac{ac - 2bd}{c^2 - 2d^2} + \sqrt{2}\frac{bc - ad}{c^2 - 2d^2}$

$\frac{\alpha}{\beta} = e' + \sqrt{2}f'$

where $e' = \frac{ac - 2bd}{c^2 - 2d^2}$, $f' = \frac{bc - ad}{c^2 - 2d^2}$

By ② $d(\delta) = d(\beta)\, d\left(\frac{\alpha}{\beta} - \gamma\right)$

$= d(\beta)\, d\left((e' + \sqrt{2}f') - (e + \sqrt{2}f)\right)$

$= d(\beta)\, d\left((e' - e) + \sqrt{2}(f' - f)\right)$

$d(\delta) = d(\beta)\left[(e' - e)^2 - 2(f' - f)^2\right]$ — ③

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We have $|e'-e| \leq \frac{1}{2}$

$|f'-f| \leq \frac{1}{2}$

By (3) $d(\delta) \leq d(\beta)\left[\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^2\right]$

$= d(\beta)\left[\frac{1}{4} - \frac{2}{4}\right]$

$= d(\beta)\left[-\frac{1}{4}\right]$

$\boxed{d(\delta) < d(\beta)}$ $\left[\because -\frac{1}{4} < 1\right]$

Hence Proved

$\therefore$ $Z[\sqrt{2}]$ is E.D

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